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Non-iterative solutions of the Bethe–Salpeter equation in a model with non-canonical propagators

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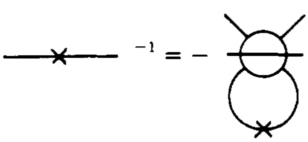
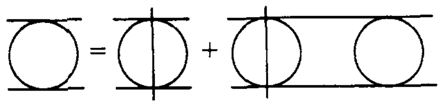
Abstract. In a planar approximation to a Yukawa-type $g\psi^*\psi\varphi$ field theory with scalar fields ψ and φ we study the Bethe-Salpeter (BS) equation for the scattering amplitude of the ψ field in the case of vanishing ψ wavefunction renormalisation constant $Z_2 = 0$. Due to the asymptotic behaviour of the non-canonical ψ propagator, given by the corresponding Dyson-Schwinger equation for $Z_2 = 0$, the Neumann series of the BS equation diverges for Euclidean values of the invariants and all masses $m^2, \mu^2 > 0$. Being responsible for this divergence, only the asymptotic part of the propagator is subsequently retained in the BS equation. Using in Euclidean metric an exactly soluble high-energy version of the BS equation and treating the difference as a perturbation, we derive a new but equivalent integral equation for the scattering amplitude. By contraction-mapping arguments we obtain existence and multiplicity results for solutions of ths transformed equation. The asymptotic behaviour of these solutions is rigorously established and found to be oscillating.

1. Introduction

The development of non-perturbative methods in quantum field theory is highly desirable in view of the deficiencies of the usual perturbation expansion, which become apparent, e.g., in the treatment of bound-state problems, spontaneous symmetry breaking and dynamical mass generation, Regge behaviour, etc. By formally summing up suitable infinite subclasses of the Feynman diagrams of a renormalisable Lagrangian quantum field theory (RQFT) one may end up with a simplified system of integral equations for Green functions. These equations often satisfy Lorentz covariance, analyticity, Lehmann spectral representation and other general requirements of QFT. In many approximation schemes the two-particle-irreducible four-point function (≡ Bethe-Salpeter kernel, hereafter symbolised by) plays an essential role as it permits the formulation of a closed system of integral equations for the lower n-point functions ($n \leqslant 4$). Specifically in the simple case of a Yukawa-type interaction of scalar fields ψ and φ with $L_1 = g\psi^*\psi\varphi$, the Dyson-Schwinger equation for the propagator and the Bethe-Salpeter (BS) equation for the scattering amplitude (symbolised by) form a coupled system of integral equations, which is completely defined once the BS kernel or some approximation of it is given. In spite of the neglect of spin and of some deficiences of the Yukawa theory, such as the presumable non-existence of a ground state, the model is expected to yield at least a qualitative description of nucleon-nucleon interaction phenomena. In graphical notation these equations are:



$+ 2Z_2 p_\mu$ (1.1)



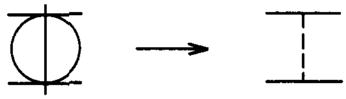
(1.2)

with

——— ≡ dressed ψ field propagator

—×— ≡ derivative WRT the momentum of the ψ line.

A simple way to derive the propagator equation (1.1) is by application of the Ward identity to the one-photon vertex function after introduction of a minimally coupled electromagnetic field A_μ [1, 2]. The simplest approximation to equations (1.1) and (1.2) is obviously the substitution of the complete BS kernel by its one-particle exchange contribution:



(1.3)

(– – – – ≡ free φ field propagator).

This approximation corresponds to the summation of ladder graphs with ψ propagators (side lines of the ladder) determined by the solutions of the one-particle exchange version of (1.1).

In contrast to previous work on similar models [3-5] we consider here only the non-canonical case of vanishing ψ wavefunction renormalisation constant $Z_2=0$. In axiomatic field theory it can be argued that under some general assumptions a non-trivial field theory will have to be a non-canonical one [6]. The same conclusion holds for field theories represented as fixed point solutions of the renormalisation group transformation [7].

From a more physical point of view the condition $Z_2=0$ has been suggested for a class of models as a criterion for the composite nature of the corresponding field (or particle) [8].

This bound-state condition has found an interesting application to bosonic bound states appearing in composite models of quarks, leptons, the intermediate vector bosons, etc, constructed in recent years in an attempt to reduce the number of fundamental particle states entering the theory [9]. We finally mention that the condition $Z_2=0$ for all fields of a given field theory is the main assumption of the so-called bootstrap hypothesis advocated by several authors [10] in an effort to abolish the difference between elementary and composite particles.

2. Summary of basic results for the propagator

Before investigating the scattering amplitude we state for later reference some basic properties of the two-point function. The Dyson–Schwinger (DS) equation for the propagator Δ of the scalar ψ field in the approximation (1.1) is given by†:

$$\Delta^{-1}(p^2)=Z_2(p^2+m^2)+\frac{\lambda}{\pi^2}\mathrm{i}\int \mathrm{d}^4k\left(\frac{1}{(p-k)^2+\mu^2-\mathrm{i}\varepsilon}-\frac{1}{(\hat{p}-k)^2+\mu^2-\mathrm{i}\varepsilon}\right)\Delta(k^2) \tag{2.1}$$

with $\lambda:=\pi^2g^2/(2\pi)^4$, $\hat{p}^2=-m^2$.

† Our metric is $(g_{\mu\nu})=(-1,1,1,1)$.

For $\mu^2 > 0$, Z_2 is fixed by the on-shell normalisation condition

$$\operatorname{res} \Delta(p^2)|_{p^2=-m^2} = 1. \tag{2.2}$$

(For $\mu^2 = 0$, off-shell normalisatsion is necessary.) Without consideration of the normalisation condition (2.2) the (DS) equation (2.1) was first treated by Saenger [11].

We consider here only the case $Z_2 = 0$, which determines a critical coupling constant $\lambda = \lambda_0$.

For $Z_2 = 0$ the DS equation (2.1) is known to have a unique solution for all masses $m^2, \mu^2 \geq 0$, which fulfils the Lehmann spectral representation [12, 13]. For $s := p^2 \geq 0$ this solution satisfies upper and lower bounds of the form [13]:

$$\Delta_1(s) \leq \Delta(s) \leq \Delta_2(s) \tag{2.3}$$

with

$$\Delta_1(s) := \frac{3}{4(2\lambda_0)^{1/2}} \frac{C_{\beta,l^2}}{(s+l^2)^{1/2}} \tag{2.4}$$

where

$$C_{\beta,l^2} := \left(1 + \frac{3}{4\beta(1-\beta)} \frac{m+\mu}{(m^2+\mu^2)^\beta} \frac{1}{(l^2)^{1/2-\beta}}\right)^{-1} \qquad (l^2 \geq m^2, 0 < \beta \leq \tfrac{1}{2})$$

and

$$\Delta_2(s) := \left(\frac{2}{\lambda_0}\right)^{1/2} \frac{1}{(s+m^2)^{1/2}} \left(1 + \frac{m+\mu}{(s+m^2)^{1/2}}\right). \tag{2.5}$$

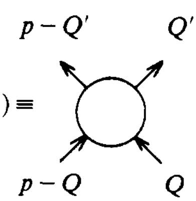
For $m^2 = 0$ the exact asymptotic behaviour of $\Delta(s)$ for $s \to \infty$ is given by (cf [12]):

$$\Delta(s) \to \left(\frac{3}{4\lambda_0}\right)^{1/2} \frac{1}{(s+l^2)^{1/2}} \equiv \Delta_{\text{as}}(s). \tag{2.6}$$

For $m^2 > 0$ we suppose that the same asymptotic relation holds but a proof of this conjecture is still missing.

3. Divergence of the Bethe–Salpeter Neumann series

We now consider the BS equation for the scattering amplitude

$T(p; Q, Q') \equiv$ 

in the one-particle exchange approximation:

$$T(p; Q, Q') = \frac{g_0^2}{p^2+\mu^2-\mathrm{i}\varepsilon} - g_0^2 \mathrm{i} \int \frac{\mathrm{d}^4 s}{(2\pi)^4} \frac{1}{(p-s)^2+\mu^2-\mathrm{i}\varepsilon} \times \Delta((s-Q)^2)\Delta((s-Q')^2) T(s; Q, Q') \tag{3.1}$$

Δ being the solution of equation (2.1).

For Euclidean external momenta (i.e. taking $p_0 = \mathrm{i}p_4$, $p_4 \in \mathbb{R}$, etc) we may Wick-rotate the integration contour in any term T_n of the formal Neumann series of the BS equation (3.1). After Wick-rotation T_n is a $4n$-fold Euclidean integral, whose integrand turns out to be *positive*. Exploiting this positivity condition, a lower bound T'_n for T_n is obtained by replacing the propagator Δ by its lower bound Δ_1 given in (2.4). Defining:

$$\begin{aligned} &\chi_n(s, p_1^2, p_2^2; \tau) := \Delta_1(p_1^2)\Delta_1(p_2^2)T'_n(s, p_1^2, p_2^2; \tau) \\ &s := p^2 \qquad p_1^2 := (p-Q)^2 \qquad p_2^2 := (p-Q')^2 \\ &t := (Q'-Q)^2 \qquad \tau := (t, Q^2, Q'^2) \end{aligned} \tag{3.2}$$

the following integral representation (sometimes called the Okubo–Feldmann representation [14]) holds for χ_n:

$$\begin{aligned} \chi_n(s, p_1^2, p_2^2; \tau) &= \int_0^\infty \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\, z^2 f_n(x, y, z; \tau) \\ &\quad \times \exp\{-z[s+\mu^2+x(p_1^2+l^2)+y(p_2^2+l^2)]\}. \end{aligned} \tag{3.3}$$

This representation has already been used in the case of canonical propagators in [3–5].

For the weight function f_n we have found the following lower bound [13]:

$$\begin{aligned} f_n(x, y, z; \tau) \geqslant & \frac{16\pi C_1}{z\sqrt{xy}}\,\theta\left(\frac{\varepsilon}{(A+1)^n} - z(1+\xi)B\right) \\ & \times \{C_1 \exp[-(1+1/A)\varepsilon]\}^n \frac{\ln^{2n}(1+\xi)}{n!(n+1)!} \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} &\xi := \min(x, y) \\ &B := \max(\mu^2, l^2+\tfrac{1}{4}t, Q^2+l^2, Q'^2+l^2) \\ &A, \varepsilon > 0 \text{ arbitrary positive real numbers.} \end{aligned}$$

The bound (3.4) for f_n implies

$$\chi_n(s, p_1^2, p_2^2; \tau) \geqslant 2G_0\left[\left(\frac{\varepsilon_1}{\varepsilon}\right)^{1/2} - 1\right]\varepsilon^2\left(\frac{\exp[-(1+1/A)\varepsilon]}{(A+1)^2}\right)^n \frac{(2n)!}{n!(n+1)!} C_1^{n+1} \tag{3.5}$$

with $\varepsilon_1 > \varepsilon$ and

$$G_0 := \frac{16\pi}{B^2}\exp\{-[\varepsilon(s+\mu^2)+\varepsilon_1(p_1^2+p_2^2+2l^2)]/B\}.$$

Since $4C_1 = \frac{9}{8}C^2_{\beta,l^2} > 1$ for β, l^2 properly chosen (cf (2.4)), the divergence of $\Sigma_n \chi_n$ follows from (3.5) by making a suitable choice of the parameters A, ε.

We remark that this divergence is due to the non-canonical asymptotic behaviour

$$\Delta(s) = \mathrm{O}(s^{-1/2}) \qquad \text{for } s \to \infty$$

of the propagator as well as the particular coefficient $\sqrt{\frac{3}{4}}$ appearing in front of $s^{-1/2}$ as given in (2.6).

If we take for Δ, e.g., the expression

$$\Delta(s)=\left(\frac{C}{\lambda_0}\right)^{1/2}\frac{1}{(s+l^2)^{1/2}}\qquad \text{with } C<\tfrac{1}{4}$$

the Neumann series of the BS equation (3.1) would be convergent in a non-empty domain of the mass and momentum variables.

4. Existence of non-iterative solutions

Since the asymptotic behaviour (2.6) of the ψ propagator Δ is responsible for the divergence of the Neumann series of the BS equation (3.1) we retain in the following only its asymptotic term

$$\Delta_{\mathrm{as}}(s)=\left(\frac{3}{4\lambda_0}\right)^{1/2}\frac{1}{(s+l^2)^{1/2}}\tag{4.1}$$

with a free mass-like parameter l^2 in the Bethe–Salpeter equation. In this way we get for Euclidean 4-vectors instead of the BS equation (3.1) the following simplified integral equation:

$$T(p;Q,Q')=T_0(p)+\frac{3}{4\pi^2}\int\frac{\mathrm{d}^4s}{(p-s)^2+\mu^2}\frac{1}{[(s-Q)^2+l^2]^{1/2}}\times\frac{1}{[(s-Q')^2+l^2]^{1/2}}T(s;Q,Q')\tag{4.2}$$

with

$$T_0(p):=g_0^2/(p^2+\mu^2).$$

Restricting ourselves to†

$$Q=-Q'\equiv\tfrac{1}{2}q$$

we obtain after angular integration in the simplified BS equation (4.2)

$$T(r,\chi;t)=T_0(r)+\int_0^\infty \mathrm{d}r'\int_0^\pi \mathrm{d}\chi'\,K(r,\chi,r',\chi';t)T(r',\chi';t)\tag{4.3}$$

with

$$T(r,\chi;t)\equiv T(p;\tfrac{1}{2}q,-\tfrac{1}{2}q)\qquad T_0(r):=g_0^2/(r+\mu^2)$$

$$t:=q^2\qquad \cos\chi:=pq/|p||q|\qquad r:=p^2$$

$$K(r,\chi,r',\chi';t):=\frac{3}{8\pi}\left(\frac{r'}{r}\right)^{1/2}\frac{\sin\chi'}{\sin\chi}\ln\left(1+\frac{4(rr')^{1/2}\sin\chi\sin\chi'}{r+r'+\mu^2-2(rr')^{1/2}\cos(\chi-\chi')}\right)\times\frac{1}{[(r'+a)^2-r't\cos^2\chi']^{1/2}}$$

$$\chi,\chi'\in[0,\pi]\qquad a:=l^2+\tfrac{1}{4}t.$$

† Under this condition only s-wave scattering occurs.

The dependence of $T(r, \chi; t)$ on the angle χ is taken into account by the following Gegenbauer expansion:

$$T(r, \chi; t) = \sum_{l=0}^{\infty} t_l(r; t) C_l^1(\cos\chi) \tag{4.4}$$

with

$$C_l^1(\cos\chi) := \sin(l+1)\chi/\sin\chi.$$

Because of the symmetry $\chi \to \pi - \chi$ of (4.3) we take $t_{2l+1} = 0$, $l = 0, 1, \ldots$. To ensure the absolute convergence of the integral in (4.3), we impose the following sufficient condition on the t_{2l}:

$$\sum_{l=0}^{\infty} \int_0^{\infty} \frac{s|t_{2l}(s; t)|}{(s+a)^2} \mathrm{d}s < \infty. \tag{4.5}$$

A solution of (4.3) of the form (4.4) satisfying condition (4.5) will hereafter be referred to as an A_0-solution of (4.3).

We remark that the set of all A_0-functions is not mapped into itself by equation (4.3). Insertion of (4.4) into (4.3) leads to the following system of integral equations for the t_{2k}:

$$t_{2k}(r; t) = T_0(r)\delta_{k0} + \sum_{l=0}^{\infty} \int_0^{\infty} K_{kl}(r, s; t) t_{2l}(s; t)\, \mathrm{d}s \qquad k = 0, 1, 2, \ldots \tag{4.6}$$

with

$$K_{kl}(r, s; t) := \frac{3}{2\pi} \frac{1}{2k+1} \frac{2^{2k+1} r^k s^k}{\{r+s+\mu^2+[(r+s+\mu^2)^2 - 4rs]^{1/2}\}^{2k+1}}$$

$$\times \frac{s}{(st)^{1/2}} [Q_{|k-l|-1/2}(2z-1) - Q_{k+l+1/2}(2z-1)]$$

$$z := (s+a)^2/st \qquad a := l^2 + t/4.$$

Equations (4.3) and (4.6) are equivalent in the sense that they have the same A_0-solutions. Using condition (4.5) and the bounds (A1.3) and (A1.4) for $Q_{k-1/2}$, it is straightforward to show that $t_{2k}(r; t)$ is continuous in r for $r > 0$ and $\forall k \in \mathbb{N}_0$ and that

$$\begin{aligned} &|t_{2k}(r; t)| \leq C_t \qquad k = 1, 2, \ldots \\ &|t_0(r; t)| \leq C_t(r+a)/r \qquad \text{with } C_t > 0 \end{aligned} \tag{4.7}$$

for any A_0-solution $(t_{2k})_{k\in\mathbb{N}_0}$ of (4.6). Since the kernels $K_{kl}(r, s; t)$ with $(k, l) \neq (0, 0)$ fall off faster than $K_{00}(r, s; t)$ for $r \to \infty$ and $s \to \infty$, we expect K_{00} to be responsible for the divergence of the Neumann series. Using for K_{00} the asymptotic expression:

$$K_{\mathrm{as}}(r, s) := \frac{3}{2} \frac{1}{r+s+|r-s|} \frac{s}{s+a} \tag{4.8}$$

we rewrite the t_0 equation of (4.6) by subtracting on both sides the asymptotically most singular term $K_{\mathrm{as}} t_0$†:

$$(1 - K_{\mathrm{as}}) t_0 = T_0 + (K_{00} - K_{\mathrm{as}}) t_0 + \sum_{l=1}^{\infty} K_{0l} t_{2l}. \tag{4.9}$$

† For any kernel $K(x, y)$ we use the notation $(Kf)(x) \equiv \int K(x, y) f(y)\, \mathrm{d}y$ for the induced integral operator K.

From (4.9) we conclude formally that the following relation holds:

$$t_0 = (1-K_{as})^{-1}T_0 + (1-K_{as})^{-1}(K_{00}-K_{as})t_0 + (1-K_{as})^{-1}\left(\sum_{l=1}^{\infty} K_{0l}t_{2l}\right). \tag{4.10}$$

Transformations of this type appear in the mathematical literature, e.g., in Krasnoselskii [15] in a general framework and in Michlin and Prössdorf [16] in the context of singular intgral equations. Cosenza *et al* [17] applied a similar technique to the case of a singular (homogeneous) Bethe-Salpeter equation to get a transformed equation of Fredholm character. This transformation enables the authors to extablish analyticity properties of the bound state condition. The existence of solutions to the transformed equation is, however, not shown.

To give a rigorous meaning to equation (4.10) we have to construct the inverse operator of $1-K_{as}$. This is done in the following

Lemma 4.1. Let $g(\cdot)$ be continuous on $(0,\infty)$ and satisfy the condition:

$$\int_0^\infty \frac{x|g(x)|}{(x+a)^{3/2}}\,dx < \infty. \tag{4.11}$$

Then:

$$(1-K_{as})\varphi = g$$

$$\Leftrightarrow \varphi(x) = A\varphi^{(0)}(x) + \int_0^\infty R_a(x,y)g(y)\,dy \qquad \text{with } A\in\mathbb{R} \tag{4.12}$$

$$\begin{aligned}
&\varphi^{(0)}(x) := \left(\frac{x+a}{x}\right)^{1/2} P_\nu^1\left(1+\frac{2x}{a}\right) \qquad \nu := -\tfrac{1}{2}+\mathrm{i}\sqrt{\tfrac{1}{2}}\\
&R_a(x,y) := \delta(x-y) + \bar{R}_a(x,y)\\
&\bar{R}_a(x,y) := \frac{2}{a}\left(\frac{x+a}{x}\right)^{1/2}\left(\frac{y}{y+a}\right)^{1/2}\\
&\left[\theta(x-y)\,\mathrm{Re}\,Q_\nu^1\left(1+\frac{2x}{a}\right)P_\nu^1\left(1+\frac{2y}{a}\right) + (x\leftrightarrow y)\right]
\end{aligned} \tag{4.13}$$

where P_ν^1, Q_ν^1 are associated Legendre functions.

Proof. Under the conditions on g stated in the lemma, the integral equation

$$(1-K_{as})\varphi = g$$

is equivalent to the following singular Sturm-Liouville boundary-value problem for the function $\psi(x) := x(\varphi(x)-g(x))$:

$$\begin{aligned}
&x(x+a)\psi''(x) + \tfrac{3}{4}\psi(x) = -\tfrac{3}{4}xg(x)\\
&\psi(0) = 0 \qquad \psi'(\infty) = 0.
\end{aligned} \tag{4.14}$$

By standard methods [18] the general solution to (4.14) is found to be:

$$\psi(x) = A\psi_1(x) + \frac{2}{a}\psi_2(x)\int_0^x \frac{\psi_1(y)}{y+a}g(y)\,dy + \frac{2}{a}\psi_1(x)\int_x^\infty \frac{\psi_2(y)}{y+a}g(y)\,dy \tag{4.15}$$

with $A\in\mathbb{R}$.

In (4.15) ψ_1, ψ_2 are a fundamental system of solutions of the homogeneous equation of (4.14), for which we may take:

$$\begin{aligned}\psi_1(x) &= [x(x+a)]^{1/2} P_\nu^1(1+2x/a)\\ \psi_2(x) &= [x(x+a)]^{1/2}\,\mathrm{Re}\, Q_\nu^1(1+2x/a)\end{aligned} \tag{4.16}$$

with $\nu := -\frac{1}{2}+\sqrt{\frac{1}{2}}$. Going back from ψ to φ, the statement of the lemma follows.

Using the inequalities for P_ν^1, Q_ν^1 given in appendix 1 we obtain the following useful bound for $\bar{R}_a$:

$$|\bar{R}_a(x,y)| \leqslant \left(\frac{3}{4}\right)^{3/2} \frac{(8+\pi^2)^{1/2}(x+a)^{1/2}}{x} \frac{y}{(y+a)^{3/2}}. \tag{4.17}$$

Application of the estimates (4.7), valid for A_0-solutions of (4.6), and of the bound (A2.3) for $|K_{00}(x,y)-K_{as}(x,y)|$ to the function

$$g \equiv T_0 + (K_{00}-K_{as})t_0 + \sum_{l=1}^{\infty} K_{0l}t_{2l}$$

implies in view of lemma 4.1 the following theorem.

Theorem 4.2. $\bar{t} = (t_{2k})_{k\in\mathbb{N}_0}$ is an A_0-solution of (4.6) if and only if $\exists A \in \mathbb{R}$ such that $t_{2k} = t_{2k}^A$ with $\bar{t}^A = (t_{2k}^A)_{k\in\mathbb{N}_0}$ an A_0-solution of the system:

$$\begin{aligned} t_{2k}^A &= \sum_{l=0}^{\infty} K_{kl} t_{2l}^A \qquad k=1,2,\ldots\\ t_0^A &= A\varphi^{(0)} + R_a T_0 + R_a(K_{00}-K_{as})t_0^A + R_a\left(\sum_{l=1}^{\infty} K_{0l}t_{2l}^A\right). \end{aligned} \tag{4.18}$$

Heuristic considerations of system (4.18) lead us to expect an inequality of the form

$$|t_{2k}(r;t)| \leqslant C\frac{r^{k-1}t^k}{(r+a)^{2k-1}} \qquad C>0$$

to be valid for $k=1,2,\ldots$ for any solution t_{2k} of (4.18). For t_0 it can be shown that

$$\sup_{r>0} \frac{r}{a^{1/2}(r+a)^{1/2}} |t_0(r;t)| < \infty$$

for A_0-solutions $\bar{t} = (t_0, t_2, t_4, \ldots)$.

This motivates the following definitions. For $\bar{t} := (t_0, t_2, t_4, \ldots)$ with t_{2k} continuous on $(0,\infty)$ we define:

$$F_i := \{\bar{t} \mid \|\bar{t}\|_i < \infty\}$$

with

$$\|\bar{t}\|_i := \sup_{\substack{x>0\\ k\in\mathbb{N}_0}} g_k^{(i)}(x)|t_{2k}(x)| \qquad i=1,2 \tag{4.19}$$

and the weight functions:

$$g_0^{(1)}(x) := g_0^{(2)}(x) := \frac{x}{a^{1/2}(x+a)^{1/2}}$$
$$g_k^{(1)}(x) := \frac{(x+a)^{2k-1}}{x^{k-1}t^k} \qquad k=1,2,\ldots \tag{4.20}$$
$$g_k^{(2)}(x) := 1 \qquad k=1,2,\ldots.$$

One obviously has $F_1 \subset F_2$.

Every A_0-solution of (4.18) is contained in F_2 as can be seen by means of the estimates (4.7) for t_{2k}, $k \geq 1$ and (4.17) for $\bar{R}_a$ applied to the t_0 equation of (4.18).

We now define an operator $\bar{K}$, which acts on the sequences $\bar{t}$, in the following way:

$$\bar{K}: \begin{cases} t_{2k} \mapsto t'_{2k} := \sum_{l=0}^{\infty} K_{kl} t_{2l} & k \geq 1 \\ t_0 \mapsto t'_0 := R_a(K_{00} - K_{as}) t_0 + R_a \sum_{l=1}^{\infty} K_{0l} t_{2l}. \end{cases} \tag{4.21}$$

We consider the operator $\bar{K}$ on the spaces F_1 and F_2.

Lemma 4.3. $\bar{K}$ is a bounded operator in the Banach spaces F_1 and F_2. The respective norms satisfy the following estimates.

(i) $$\|\bar{K}\|_1 \leq \max(d^{(1)}(\gamma, \varepsilon), D^{(1)}(\varepsilon)) \tag{4.22}$$

with $\gamma := l^2/\mu^2$, $\varepsilon := t/l^2$,

$$d^{(1)}(\gamma, \varepsilon) := d_0(\gamma) + c_1 \varepsilon (1 + \tfrac{1}{4}\varepsilon)^{-3/4} [1 + \varepsilon/(4+\varepsilon)] \tag{4.23}$$
$$d_0(\gamma) := \gamma^{-1}[a_1 + a_2 \ln(1+\gamma^{1/2})] \tag{4.24}$$
$$D^{(1)}(\varepsilon) := \tfrac{3}{8}(1+\tfrac{1}{4}\varepsilon)^{1/4}[2 + \varepsilon^2/(4+\varepsilon)^2] \tag{4.25}$$

the constants have the values $a_1 := 3.609\ldots$, $a_2 := 11.237\ldots$ and $c_1 := 4.045\ldots.$

(ii) $$\|\bar{K}\|_2 \leq \max(d^{(2)}(\gamma, \varepsilon), D^{(2)}(\varepsilon)) \tag{4.26}$$

with

$$d^{(2)}(\gamma, \varepsilon) := d_0(\gamma) + c_2 \varepsilon (1 + \tfrac{1}{4}\varepsilon)^{-3/4} \tag{4.27}$$
$$D^{(2)}(\varepsilon) := \tfrac{3}{8}(1+\tfrac{1}{4}\varepsilon)^{1/4}[1 + 6\varepsilon/(4+\varepsilon)] \tag{4.28}$$

where $c_2 := 11.659\ldots.$ The proof of lemma 4.3 is given in appendix 2.

Applying the contraction mapping principle to the system (4.18), the following theorem is an immediate consequence of lemma 4.3.

Theorem 4.4. (i) Let $\gamma \geq 23.5$, $\varepsilon < (1 - d_0(\gamma))/4.12$ ($\equiv$ domain $\mathscr{B}_1$). Then for $\forall A \in \mathbb{R}$ system (4.18) has in F_1 exactly one A_0-solution $\bar{t}^A = (t_{2k}^A)_{k \in \mathbb{N}_0}$.

(ii) For $\gamma \geq 23.5$, $\varepsilon < (1 - d_0(\gamma))/11.66$ ($\equiv$ domain $\mathscr{B}_2$), no further A_0-solutions of (4.18) (other than those stated under (i)) exist. In this case the one-parameter family

$$T_A(r, \chi; t) = \sum_{l=0}^{\infty} t_{2l}^A(r; t) C_{2l}^1(\cos\chi) \qquad A \in \mathbb{R}$$

contains all A_0-solutions of (4.2).

5. Asymptotic behaviour

In the Bjorken limit $r \to \infty$ we specify the exact asymptotic behaviour of the solutions given in theorem 4.4. This limit is of particular importance in the analysis of deep-inelastic electron-hadron scattering [1].

Theorem 5.1. Let $(\gamma, \varepsilon) \in \mathscr{B}_1$ and $T = T_A$ for $A \in \mathbb{R}$ be the A_0-solution of the simplified BS equation (4.2) as specified in theorem 4.4(i). Then for $r \to \infty$:

$$T_A(r, \chi; t) = \left(\frac{a}{\pi r}\right)^{1/2} \left[A_1 \sin\left(\frac{1}{\sqrt{2}} \ln \frac{4r}{a}\right) + A_2 \cos\left(\frac{1}{\sqrt{2}} \ln \frac{4r}{a}\right)\right] + \mathrm{O}(1/r). \tag{5.1}$$

with

$$A_1 := \frac{\alpha C}{a} \tan\left(\frac{\pi}{\sqrt{2}}\right) - \beta A \qquad A_2 := \alpha A + \frac{\beta C}{a} \tan\left(\frac{\pi}{\sqrt{2}}\right)$$

$$\alpha + \mathrm{i}\beta := \Gamma(\mathrm{i}\sqrt{\tfrac{1}{2}})/\Gamma(-\tfrac{1}{2} + \mathrm{i}\sqrt{\tfrac{1}{2}})$$

$$C := \int_0^\infty \left(\frac{x}{x+a}\right)^{1/2} P_\nu^1\left(1 + \frac{2x}{a}\right) g(x)\, \mathrm{d}x \qquad \nu = -\tfrac{1}{2} + \mathrm{i}\sqrt{\tfrac{1}{2}} \tag{5.2}$$

$$g := T_0 + (K_{00} - K_{\mathrm{as}}) t_0^A + \sum_{l=1}^{\infty} K_{0l} t_{2l}^A \tag{5.3}$$

provided that A_1 and A_2 do not vanish simultaneously.

Proof. Since $\bar{t} \in F_1$ for the solution $\bar{t} = \bar{t}^A$ of (4.18) considered here, we have for $T_R := T_A - t_0$ the estimate:

$$|T_R(r, \chi; t)| \leq \sum_{l=1}^{\infty} (2l+1) \frac{r^{l-1} t^l}{(r+a)^{2l-1}} \|\bar{t}\|_1 \leq \frac{3t}{r+a} (1 + \tfrac{1}{4}\varepsilon)^2 \|\bar{t}\|_1 .$$

This shows that t_0 is the leading term of T_A for $r \to \infty$ if t_0 falls off less fast than $1/r$. Taking account of (4.18), we conclude that t_0 satisfies the equation:

$$t_0 = A\varphi^{(0)} + g + \bar{R}_a g$$

with g as in (5.3). For $r \to \infty$ we have: $g(r) = \mathrm{O}(1/r)$. This implies:

$$(\bar{R}_a g)(r) = \frac{2C}{a} \left(\frac{r+a}{r}\right)^{1/2} \operatorname{Re} Q_\nu^1\left(1 + \frac{2r}{a}\right) + \mathrm{O}(1/r)$$

with C given by (5.2). The desired result now follows from the well known asymptotic behaviour of P_ν^1 and Q_ν^1 [19].

For $A = 0$ one can show that [13]:

$$C < 0 \qquad \text{for } \gamma \geq 86.5,\ \varepsilon < (1 - d_0(\gamma))(1 - \delta(\gamma))/12.8$$

with

$$\delta(\gamma) := \frac{3}{2}\left(1 + \frac{8}{\pi^2}\right)^{1/2} \frac{1}{\gamma} \left[\ln(1 + \sqrt{\gamma}) + \frac{3}{4} \frac{B}{1 - d_0(\gamma)} \left(\frac{\pi}{4} + 2 \ln(1 + \sqrt{\gamma})\right)\right]$$

where $B := 6.491\ldots$ and d_0 is as given in (4.24).

We remark that the oscillatory behaviour of all solutions of the simplified BS equation (4.2) for $Q = Q' = 0$, $\gamma \geqslant 86.5$, implies the divergence of the Neumann series of this equation in the above γ range, thus confirming the results of § 3.

6. Conclusions and outlook

By means of a transformation technique we have converted in Euclidean metric the BS equation of a specific model with divergent Neumann series into an equivalent integral equation, which defines a contractive mapping in a suitable Banach space. Having thus solved the existence and multiplicity problem, the asymptotic behaviour of the solutions in the Bjorken limit could then be exactly specified. This result is due to the fact that the above-mentioned transformation extracts the asymptotically dominant part from the integral kernel. We expect this technique to be applicable also in many other models.

Treating the Regge limit instead of the Bjorken limit is much more involved since this limit invokes not only Euclidean 4-vectors. We have so far not succeeded in giving the exact leading term in this limit.

We finally remark that our model has the unpleasant property of not providing a unique solution for the off-shell scattering amplitude. One may try to overcome or at least to reduce this non-uniqueness by imposing some additional requirements on the scattering amplitude, such as, e.g., the validity of dispersion relations. Whether any of the given solutions will fulfil such relations with correct threshold properties is, however, not known and remains to be investigated.

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Appendix 1. Inequalities for Legendre functions

We list some elementary estimates of Legende functions which we have used in our calculations but which do not occur in the standard literature. As they follow in an obvious manner from the corresponding integral representations, we give no proof here.

We have for $x > 0$, $\nu := -\frac{1}{2} + \mathrm{i}\sqrt{\frac{1}{2}}$

$$\left| P_\nu^1\left(1 + \frac{2x}{a}\right)\right| \leqslant \frac{3}{4}\left(1 + \frac{8}{\pi^2}\right)^{1/2} \frac{(ax)^{1/2}}{x + a} \tag{A1.1}$$

$$\left| Q_\nu^1\left(1 + \frac{2x}{a}\right)\right| \leqslant \frac{\pi}{2}\left(\frac{3}{4}\frac{a}{x}\right)^{1/2} \tag{A1.2}$$

and for $z>1$, $k=1,2,\ldots$ we have

$$Q_{k-1/2}(2z-1) \leqslant z^{-k} Q_{-1/2}(2z-1) \tag{A1.3}$$

$$Q_{-1/2}(2z-1) \leqslant \frac{\pi}{2}(1+\tfrac{1}{4}\varepsilon)^{1/4} \frac{(yt)^{1/2}}{y+a} \tag{A1.4}$$

where the definitions

$$z := (y+a)^2/yt \qquad \varepsilon := t/l^2 \qquad a := l^2 + t/4$$

have been used in (A1.4).

Appendix 2. Proof of lemma 4.3

Due to limitations of space, we omit technical details, which the reader can find in [13].

(i) The estimate

$$\|\bar{K}\|_1 \leqslant \max(N_0^{(1)}, N_1^{(1)}) \tag{A2.1}$$

is obviously valid if we define $N_0^{(1)}$ and $N_1^{(1)}$ as follows:

$$N_0^{(1)} := n_0 + n_1 + n_2^{(1)}$$

with

$$n_i := \sup_{x>0} \frac{x}{a^{1/2}(x+a)^{1/2}} k_i(x) + (\tfrac{3}{4})^{3/2}(8+\pi^2)^{1/2} \int_0^\infty \mathrm{d}y \frac{y}{a^{1/2}(y+a)^{3/2}} k_i(y) \qquad i=0,1, \tag{A2.2}$$

and similarly for $n_2^{(1)}$ and $k_2^{(1)}$.

In (A2.2) we have made use of the bound (4.17) of $\bar{R}_a(x,y)$, the estimates (A1.3) and (A1.4) for $Q_{k-1/2}$, and the inequality

$$|K_{00}(x,y) - K_{\mathrm{as}}(x,y)| \leqslant K_0(x,y) + K_1(x,y) \tag{A2.3}$$

with

$$K_0(x,y) := \frac{3}{2} \frac{\mu^2}{x+y+|x-y|} \frac{1}{|x-y|+\mu^2} \frac{y}{y+a}$$

$$K_1(x,y) := \tfrac{3}{2}(1+\tfrac{1}{4}\varepsilon)^{1/4} \frac{t}{x+y+|x-y|} \frac{y^2}{(y+a)^3}.$$

k_0, k_1 and $k_2^{(1)}$ are the contributions of K_0, K_1 and of $\Sigma_{l=1}^{\infty} K_{0l} t_{2l}$, respectively, in the t_0 equation of (4.18) to the norm $\|\bar{K}\|_1$. Explicitly we have the following definitions and estimates:

$$k_0(x) := \tfrac{3}{2}\mu^2 \int_0^\infty \frac{1}{x+y+|x-y|} \frac{1}{|x-y|+\mu^2} \left(\frac{a}{y+a}\right)^{1/2} \mathrm{d}y$$

$$\leqslant \frac{3}{2} \frac{\mu^2 a^{1/2}}{x(x+a)^{1/2}} \left\{ \frac{1}{2} \ln\left(1+\frac{x}{\mu^2}\right) + \ln\left[1+\left(\frac{x}{\mu^2}\right)^{1/2}\right] + \frac{x}{x+a} \right\} \tag{A2.4}$$

$$k_1(t) := \tfrac{3}{2}\left(1+\tfrac{1}{4}\varepsilon\right)^{1/4} \int_0^\infty \frac{1}{x+y+|x-y|} \frac{a^{1/2} t y}{(y+a)^{5/2}}\, \mathrm{d}y$$

$$\leqslant \left(1+\tfrac{1}{4}\varepsilon\right)^{1/4} \frac{t}{x+a} \tag{A2.5}$$

$$k_2^{(1)}(x) := \sum_{l=1}^{\infty} \int_0^\infty K_{0l}(x, y; t) \frac{t^l y^{l-1}}{(y+a)^{2l-1}}\, \mathrm{d}y$$

$$\leqslant \tfrac{1}{4}\left(1+\tfrac{1}{4}\varepsilon\right)^{1/4} \frac{t^2}{a(x+a)}. \tag{A2.6}$$

Inserting these estimates into (A2.2) we obtain the following bounds for n_0, n_1 and $n_2^{(1)}$:

$$n_0 \leqslant \gamma^{-1}[a_1 + a_2 \ln(1+\sqrt{\gamma})] \equiv d_0(\gamma) \tag{A2.7}$$

with $a_1 = 3.609\ldots$, $a_2 = 11.237\ldots$.

$$n_1 \leqslant c_1 \varepsilon \left(1+\tfrac{1}{4}\varepsilon\right)^{-3/4} =: d_1(\varepsilon) \tag{A2.8}$$

$$n_2^{(1)} \leqslant \tfrac{1}{4} c_1 \varepsilon^2 \left(1+\tfrac{1}{4}\varepsilon\right)^{-7/4} =: d_2^{(1)}(\varepsilon) \tag{A2.9}$$

with $c_1 := 4.045\ldots$.

Combining these estimates yields the following inequality for $N_0^{(1)}$:

$$N_0^{(1)} \leqslant d_0(\gamma) + d_1(\varepsilon) + d_2^{(1)}(\varepsilon)$$

$$= d_0(\gamma) + c_1 \varepsilon \left(1+\tfrac{1}{4}\varepsilon\right)^{-3/4} \left(1 + \frac{\varepsilon}{4+\varepsilon}\right)$$

$$=: d^{(1)}(\gamma, \varepsilon). \tag{A2.10}$$

$N_1^{(1)}$, which takes into account the contribution to $\|\bar{K}\|_1$ of the t_{2k} equations of (4.18) for $k \geqslant 1$, is defined and estimated as follows:

$$N_1^{(1)} := \sup_{\substack{x>0\\ k\in\mathbb{N}}} \frac{(x+a)^{2k-1}}{x^{k-1} t^k} \sum_{l=0}^{\infty} \int_0^\infty K_{kl}(x, y; t) \frac{y^{l-1} t^l}{(y+a)^{2l-1}}\, \mathrm{d}y$$

$$\leqslant \tfrac{3}{8}\left(1+\tfrac{1}{4}\varepsilon\right)^{1/4} \left[2 + \left(\frac{\varepsilon}{4+\varepsilon}\right)^2\right]$$

$$=: D^{(1)}(\varepsilon). \tag{A2.11}$$

This proves part (i) of the lemma.

(ii) The estimate

$$\|\bar{K}\|_2 \leqslant \max(N_0^{(2)}, N_1^{(2)}) \tag{A2.12}$$

holds if we define

$$N_0^{(2)} := n_0 + n_1 + n_2^{(2)} \tag{A2.13}$$

with

$$n_2^{(2)} := \sup_{x>0} \frac{x}{a^{1/2}(x+a)^{1/2}} k_2^{(2)}(x) + \left(\tfrac{3}{4}\right)^{3/2}(8+\pi^2)^{1/2} \int_0^\infty \mathrm{d}y \frac{y}{a^{1/2}(y+a)^{3/2}} k_2^{(2)}(y) \tag{A2.14}$$

and

$$k_2^{(2)}(x) := \sum_{l=1}^{\infty} \int_0^{\infty} K_{0l}(x, y; t)\, \mathrm{d}y$$
$$\leqslant \tfrac{3}{4}(1+\tfrac{1}{4}\varepsilon)^{1/4} \frac{t}{x}\left[\ln\left(1+\frac{x}{a}\right) - \frac{1}{2}\frac{x}{x+a}\right]. \tag{A2.15}$$

This implies that

$$n_2^{(2)} \leqslant c_1' \varepsilon (1+\tfrac{1}{4}\varepsilon)^{-3/4} =: d_2^{(2)}(\varepsilon) \tag{A2.16}$$

with $c_1' := 7.614\ldots$, and

$$N_0^{(2)} \leqslant d_0(\gamma) + c_2 \varepsilon (1+\tfrac{1}{4}\varepsilon)^{-3/4} =: d^{(2)}(\gamma, \varepsilon) \tag{A2.17}$$

with $c_2 := 11.659\ldots$.
The definition and upper bound of $N_1^{(2)}$ are given by

$$N_1^{(2)} := \sup_{\substack{x>0 \\ k\in\mathbb{N}}} \left(\sum_{l=1}^{\infty} \int_0^{\infty} K_{kl}(x, y; t)\, \mathrm{d}y + \int_0^{\infty} K_{k0}(x, y; t) \frac{a^{1/2}(y+a)^{1/2}}{y}\, \mathrm{d}y \right)$$
$$\leqslant \tfrac{3}{8}(1+\tfrac{1}{4}\varepsilon)^{1/4}\left(1+\frac{6\varepsilon}{4+\varepsilon}\right)$$
$$=: D^{(2)}(\varepsilon). \tag{A2.18}$$

This completes the proof of lemma 4.3.

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